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On Complex Strictly Convex Spaces, I

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1. INTRODUCTION

An important and interesting class of Banach spaces was introduced by Clarkson under the name "strictly convex spaces" which are all normed spaces such that each point of the unit sphere is an extreme point of the unit ball.

In the study of analytic functions on Banach spaces, in connection with the validity of the strong maximum modulus theorem an interesting generalization of the notion of extreme point was very useful. This generalization considered by E. Thorp and R. J. Whitley [11] permits a characterization of all Banach spaces in which the strong maximum modulus theorem holds.

The consideration of "complex extreme points" by E. Thorp and R. J. Whitley suggests us to consider a class of Banach spaces which we call "complex strictly convex" and the purpose of the present note is to give some properties and examples.

2. COMPLEX STRICTLY CONVEX SPACES

First we recall the definition of complex extreme points as was given by Thorp and Whitley.

DEFINITION 2.1. Let C be a convex subset of a complex Banach space X and $x_0 \in C$. The element x_0 is a complex extreme point of C if $\{x_0 + zy \colon |z| \leqslant 1\} \subset C$ for $y \in X$ then $y = 0$.

It is easy to see that we can suppose only z, $|z| = 1$.

It is easy to see that every extreme point of a convex set in a complex space is a complex extreme point.

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The notion of "complex strictly convex spaces" is introduced in

DEFINITION 2.2. A complex Banach space X is called complex strictly convex if each point of the unit sphere is a complex extreme point of the unit ball.

From the above remark on extreme points it is clear that every strictly convex space is complex strictly convex space.

The following simple theorem is useful for examples of complex strictly convex spaces.

THEOREM 2.1. *If X is a complex strictly convex Banach space then every closed subspace is also complex strictly convex space.*

Proof. Clear.

From this theorem we obtain another example of complex strictly convex space.

Indeed in [11] it is proved that every point on the unit sphere of $L^1(\Omega, B, \mu)$ is a complex extreme point. Thus $L^1(\Omega, B, \mu)$ is complex strictly convex space. Let $\Gamma = \{z, \mid z \mid = 1\}$ and $\mu = (1/2\pi)\, d\theta$ and from the above theorem we obtain that the Hardy space H^1 is also complex strictly convex.

We conjecture at this point that H^∞ and $A(\Gamma) = \{f, f$ continuous on Γ and $c_{-k}(f) = 0$ for $k = 1, 2, 3,...\}$ are also complex strictly convex spaces. In the following we give some characterization of complex strictly convex spaces using the notion of semi-inner product of Lumer [9]. We recall that a semi-inner product on a complex Banach space is a map from $X \times X$ with values in C such that

(1) $[\lambda x + y, \mu z] = \lambda\bar{\mu}[x, z] + \bar{\mu}[y, z]$,

(2) $[x, x] = \| x \|^2$,

(3) $|[x, y]| \leqslant \| x \| \| y \|$.

In [9] Lumer has shown that in any normed linear space X, one can construct a semi-inner product [,] and there may be more than one.

THEOREM 2.4. *The complex Banach space X is complex strictly convex if and only if*

$$\| u + e^{i\theta} v \| \leqslant \| u \| \qquad \textit{and} \qquad [v, u] = 0 \Rightarrow v = \theta.$$

Proof. We can suppose without loss of generality that $\| u \| = 1$. If X is complex strictly convex and

$$\| u + e^{i\theta} v \| \leqslant 1 \qquad \text{and} \qquad [v, u] = 0$$

then for all z, $|z| \leqslant 1$ we have that

$$\|u\|^2 = 1 = [u, u] = [u + zv, u] \leqslant \|u + zv\|$$
$$= \|re^{i\theta}v + re^{i\theta}(e^{-i\theta}u) + (1-r)\, e^{i\theta}(e^{-i\theta}u)\| \leqslant r\,\|v + e^{i\theta}u\| + (1-r)\,\|u\| \leqslant 1$$

which gives that $u + zv \in S(\theta, 1)$. This implies $v = \theta$.

For the converse, we suppose that X is not complex strictly convex and thus for some x_0, $\|x_0\| = 1$ and some $y_0 \in X$ and all z, $|z| \leqslant 1$

$$\|x_0 + zy_0\| \leqslant 1.$$

We show that $[y_0, x_0] = 0$. Indeed in the contrary case we have

$$|[x_0 + zy_0, x_0]| = |z[y_0, x_0] + 1| \leqslant 1$$

which clearly represents a contradiction since z is arbitrary. Thus $[y_0, x_0] = 0$. But this implies $y_0 = 0$, a contradiction. The theorem is proved.

In the same manner we can prove the following theorem

THEOREM 2.5. *The complex Banach space is complex strictly convex if and only if*

(1) $\|u + zv\| = 1 \qquad \forall z, \ |z| \leqslant 1,$

(2) $\|u\| = 1,$

(3) $[v, u] = 0,$

imply $v = \theta$.

3. Complex Extreme Points and Banach Algebras

Let $\mathscr{A}$ be a Banach algebra with unit e. We recall an important fact proved by Bohnenblust and Karlin: the unit e is a vertex of $\mathscr{A}$. (It is known that any vertex is an extreme point.)

We prove the following

THEOREM 3.1. *Every unitary element u of $\mathscr{A}$ is a complex extreme point.*

Proof. If for some $v \in \mathscr{A}$ we have

$$\|u + zv\| \leqslant 1 \qquad \forall z, \quad |z| \leqslant 1,$$

then

$$\|e + zu^{-1}v\| \leqslant 1$$

which gives $v = \theta$.

Remarks. (1) It is of interest if u is in fact a vertex of $\mathscr{A}$.

(2) The theorem is of some interest especially in the case of C^*-algebras.

4. Product Spaces of Complex Strictly Convex Spaces

Consider X and Y be two complex strictly convex spaces and the problem is about the nature of the product space $X \times Y$. We have the following

THEOREM 4.1. *If X, Y are two complex strictly convex spaces then the product space $X \times Y$ with the norm*

$$\|(x, y)\| = (\| x \|^2 + \| y \|^2)^{1/2}$$

is also complex strictly convex.

Proof. We use Theorem 2.5. For this we remark that if $[\ ,\ ]_X$, $[\ ,\ ]_Y$ are the semi-inner products on X and Y, respectively, then a semi-inner product on $X \times Y$ is

$$[(x, y), (\tilde{x}, \tilde{y})] = [x, \tilde{x}] + [y, \tilde{y}].$$

Suppose now that we have $(u, v) \in X \times Y$ such that

(1) $\| u + zv \| = 1, \qquad | z | \leqslant 1,$

(2) $\| u \| = 1,$

(3) $[v, u] = 0.$

If $u = (x_0, y_0)$, $v = (x, y)$ then

$$\begin{aligned} 1 &= (\| x_0 \|^2 + \| y_0 \|^2)^{1/2} \\ &= [x_0 + zx, x_0] + [y_0 + zy, y_0] \leqslant \| x_0 + zx \| \| x_0 \| + \| y_0 + zy \| \| y_0 \| \\ &\leqslant (\| x_0 + zx \|^2 + \| y_0 + zy \|^2)^{1/2} (\| x_0 \|^2 + \| y_0 \|^2)^{1/2} = 1 \end{aligned}$$

and thus

$$\begin{aligned} 1 &\leqslant \| x_0 + zx \| \| x_0 \| + \| y_0 + zy \| \| y_0 \| \\ &\leqslant (\| x_0 + zx \|^2 + \| y_0 + zy \|^2)^{1/2} (\| x_0 \|^2 + \| y_0 \|^2)^{1/2} = 1 \end{aligned}$$

and by the well-known theorem on Hölder inequality (since we have in fact an equality) there exists a constant k such that

$$\| x_0 + zx \| = k \| x_0 \|,$$
$$\| y_0 + zy \| = k \| y_0 \|.$$

Then (1) implies that $k = 1$ and since X and Y are complex strictly convex we obtain that $x/\| x_0 \| = \theta$, $y/\| y_0 \| = \theta$ and thus the theorem.

5. Quotient Spaces of Complex Strictly Convex Spaces

If X is a Banach space and L a closed linear subspace of X the quotient space X/L is a Banach space whose points are the various translates of L in X with the norm

$$\| x + L \| = \inf_{y \in L} \| x + y \| .$$

G. Köthe's problem [8] in the case of strictly convex space is as follows: To what extent is the strictly convex property of X transmited to its quotient spaces?

Results in this direction which substantiates a conjecture of Day were obtained by Day and Klee. Also in [7] a positive result is also established.

Of course Köthe's problem has a variant in our setting: To what extent is the complex strictly convex property of X transmitted to its quotient spaces?

In the following, using some result of Klee (or one adapted from him) we can give

Theorem 5.1. *If X is a complex strictly convex space and L is a reflexive subspace of X then X/L is complex strictly convex space.*

The proof is a consequence of some result stated as

Proposition 5.2. *For normed linear spaces X and Y with open unit cells U and V, respectively, the following assertions are equivalent:*

(i) *There is a closed subspace L of X such that Y is equivalent to the quotient space X/L;*

(ii) *there is a linear transformation of X onto Y such that $TU = V$.*

Proof. This is Proposition 3.1 of [7].

Proposition 5.3. *If X and Y are Banach spaces and $T: X \to Y$ a linear bounded transformation then if C is a complex strictly convex set, i.e., every point $x \in \partial C$ is a complex extreme point then $C_1 = TC$ has the same property.*

Proof. Let $x_1 \in C_1$, $x_1 \in \partial C_1$ and suppose that for some $y \in Y$ and all z, $| z | \leqslant 1$ the point $x_1 + zy \in C_1$. Since $x_1 \in TC$ we find $\tilde{x}_1 \in \partial C$ such that $T\tilde{x}_1 = x_1$ and also we find $\tilde{y}$ such that $T\tilde{y} = y$. Clearly $T(\tilde{x}_1 + z\tilde{y}) = x_1 + zy \in C_1$ and thus $\tilde{x}_1 + z\tilde{y} \in C$ which gives that $y = \theta$. But then $y = \theta$ and the proposition is proved.

Proof of Theorem 5.1. If U denotes the open unit cells of X and V of X/L, respectively, then by a remark of Klee [7], if T is the canonical map of X onto X/L, then

$$T(\mathrm{cl}\, U) = \mathrm{cl}\, V$$

and from Proposition 5.2 the conclusion follows.

6. Relative Complex Extreme Points

V. Klee has introduced the notion of relative extreme point in [7] and our goal here is to give the corresponding notion.

Definition 6.1. If C is a convex subset of a Banach space and C_1 is a convex subset of C then a point $x_0 \in C$ is a complex extreme point of C relative to C_1 if for some y, X and all z, $|z| \leqslant 1$,

$$x_0 + zy \in C_1 ,$$

then $y = \theta$.

Remark. It is clear that for $C = C_1$, x_0 is a complex extreme point if and only if it is in the sense of the above definition.

Theorem 6.2. *Every extreme point of C relative to C_1 is complex extreme point of C relative to C_1.*

Proof. Indeed, if x_0 is not a complex extreme point of C relative to C_1 then we choose $y_0 \in X$ such that

(1) $y_0 \neq \theta$,

(2) for all z, $zy_0 + x_0 \in C_1$.

Take $z = -1$ and thus x_0 is on the segment which has endpoints in $(x_0 - y_0)$ and $(x_0 + y_0)$ which is a contradiction. From Theorem 3 of [7] (see also [10]) we have the existence of a complex extreme point of C relative to C_1.

7. Some Problems

There are several problems which appear in connection with complex strictly convex spaces. These are inspired by the corresponding result for strictly normed (Banach) spaces.

Problem 7.1. Is a sufficient condition for a Banach space X to be a complex strictly convex space that every two-dimensional quotient subspace of X be complex strictly convex?

Problem 7.2. Is X^* complex strictly convex if and only if every two-dimensional quotient space of X is complex strictly convex?

Problem 7.3. If X is a reflexive space then is it complex strictly convex?

Problem 7.4. If X^{****} is complex stricty convex space then is X reflexive?

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